Vector Fields and Line Integrals: Work, Circulation, and Flux.

16.2.3 Find the gradient field of the function, $g(x,y,z) = e^{3z} - \ln(x^2+3y^2)$

Evaluate each partial derivative separately.

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(e^{3z} - \ln(x^2+3y^2)\right) \qquad \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(e^{3z} - \ln(x^2+3y^2)\right) \qquad \frac{\partial g}{\partial z} = \frac{\partial}{\partial z}\left(e^{3z} - \ln(x^2+3y^2)\right)$$

$$= 0 - \frac{1}{(x^2+3y^2)} \cdot \frac{\partial}{\partial x}(x^2+3y^2)$$

chain rule $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

$$= 0 - \frac{1}{(x^2+3y^2)} \cdot \frac{\partial}{\partial y}(x^2+3y^2)$$

$$= e^{3z} \cdot \frac{\partial}{\partial z}(3z) - 0$$

chain rule $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

$$= -\frac{2x}{(x^2+3y^2)} \qquad = -\frac{6y}{(x^2+3y^2)} \qquad = 3e^{3z}$$

Combine the results to find the gradient field.

$$\nabla g = -\frac{2x}{(x^2+3y^2)}\,i - \frac{6y}{(x^2+3y^2)}\,j + 3e^{3z}\,k$$

16.2.4 Find the gradient field of the function, $g(x,y,z) = 9xy + 5yz + 9xz$

$$\nabla g = \frac{\partial g}{\partial x}i + \frac{\partial g}{\partial y}j + \frac{\partial g}{\partial z}k$$

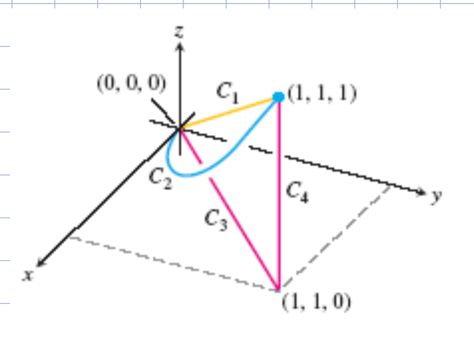
$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(9xy + 5yz + 9xz) = (9y + 9z)$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(9xy + 5yz + 9xz) = (9x + 5z)$$

$$\frac{\partial g}{\partial z} = \frac{\partial}{\partial z}(9xy + 5yz + 9xz) = (5y + 9x)$$

$$\nabla g = (9y+9z)\,i + (9x+5z)\,j + (5y+9x)\,k$$

16.2.7 Find the line integrals of $F = yi + 4xj + 2zk$ from $(0,0,0)$ to $(1,1,1)$ over each of the following paths.



$$\int_C (F\cdot T)\,ds = \int_C \left(F\cdot\frac{dr}{ds}\right)ds = \int_C F\cdot dr$$

a. The straight-line path C_1: $r(t) = ti + tj + tk$, $0 \le t \le 1$.

Vector function $F = yi + 4xj + 2zk$

$C1$: $r(t) = ti + tj + tk \qquad 0 \le t \le 1$

since $x = t,\ y = t,\ z = t \qquad F(r(t)) = ti + 4tj + 2tk$

$$\int_C F\cdot dr = \int_a^b \left(F(r(t))\cdot\frac{dr}{dt}\right)dt$$

$$= \int_0^1 (ti + 4tj + 2tk)\cdot(i + j + k)\,dt$$

$$\frac{dr}{dt} = \frac{d}{dt}(r(t)) = \frac{d}{dt}(ti + tj + tk) = i + j + k$$

$$= \int_0^1 (t + 4t + 2t)\,dt$$

$$= \int_0^1 7t\,dt$$

$$= \left[7\frac{t^2}{2}\right]_0^1$$

$$= \frac{7}{2}$$

b. The curved path $C_2: r(t) = t\,i + t^2 j + t^4 k, \ 0 \le t \le 1$.

Vector function $F = yi + 4xj + 2zk$

$C_2: r(t) = ti + t^2 j + t^4 k \quad 0 \le t \le 1$

since $x = t,\ y = t^2,\ z = t^4 \qquad F(r(t)) = t^2 i + 4tj + 2t^4 k$

$$\frac{dr}{dt} = \frac{d}{dt}(r(t)) = \frac{d}{dt}(ti + t^2 j + t^4 k) = i + 2tj + 4t^3 k$$

$$\int_C F \cdot dr = \int_a^b \left(F(r(t)) \cdot \frac{dr}{dt}\right) dt$$

$$= \int_0^1 (t^2 i + 4tj + 2t^4 k) \cdot (i + 2tj + 4t^3 k)\, dt$$

$$= \int_0^1 (t^2 + 8t^2 + 8t^7)\, dt$$

$$= \int_0^1 9t^2 + 8t^7\, dt$$

$$= [3t^3 + t^8]_0^1$$

$$= 4$$

c. The path $C_3 \cup C_4$ consisting of the line segment $(0,0,0)$ to $(1,1,0)$ followed by the segment $(1,1,0)$ to $(1,1,1)$. $\quad F = yi + 4xj + 2zk$

For C_3 the parameterization of path from $(0,0,0)$ to $(1,1,0)$ gives:

$$r(t) = (0 + (1-0)t)i + (0 + (1-0)t)j + (0 + (0-0)t)k$$
$$= ti + tj + 0k$$

since $x = t,\ y = t,\ z = 0 \qquad F(r(t)) = ti + 4tj + 2 \cdot 0k$

$$= \int_0^1 (ti + 4tj + 0k) \cdot (i + j + 0)\, dt$$

$$\frac{dr}{dt} = \frac{d}{dt}(r(t)) = \frac{d}{dt}(ti + tj + 0k) = i + j + 0$$

$$= \int_0^1 (t + 4t + 0)\, dt$$

$$= \int_0^1 5t\, dt$$

$$= \left[5\frac{t^2}{2}\right]_0^1$$

$$= \frac{5}{2}$$

For C_4 the parameterization of path from $(1,1,0)$ to $(1,1,1)$ gives:

$$r(t) = (1 + (1-1)t)i + (1 + (1-1)t)j + (0 + (1-0)t)k$$
$$= i + j + tk$$

$$\frac{dr}{dt} = \frac{d}{dt}(r(t)) = \frac{d}{dt}(i + j + tk) = 0 + 0 + 1$$

since $x = 1,\ y = 1,\ z = t \qquad F(r(t)) = i + 4j + 2tk$

$$= \int_0^1 (i + 4j + 2tk) \cdot (0 + 0 + 1)\, dt = \int_0^1 2t\, dt$$

$$= \int_0^1 (0 + 0 + 2t)\, dt = \left[2\frac{t^2}{2}\right]_0^1 = 1$$

Therefore: $C_3 + C_4$ is $\frac{5}{2} + 1 = \frac{7}{2}$

16.2.13 Find the line integral along the given path C.

$$\int_C (x-y)\,dx \,, \quad \text{where} \quad x=t \,,\ y=9t+8 \,, \quad \text{for} \quad 0 \le t \le 9$$

$x=t$ then $dx = 1\,dt$ substitute $x=t$, $y=9t+8$, and $dx=1\,dt$ and simplify.

$$\int_C (x-y)\,dx = \int_C (t-(9t+8))\,dt$$

$$= \int_C (-8t-8)\,dt$$

$$= -\int_0^9 (8t+8)\,dt$$

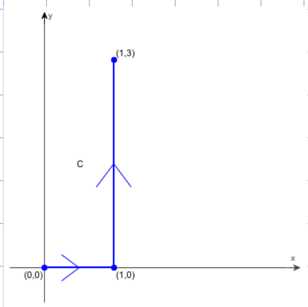
$$= -\left[4t^2+8t\right]_0^9$$

$$= -\left[4 \cdot 9^2 + 8 \cdot 9\right]$$

$$= -396$$

16.2.13 Find the line integral along the given path C.

$\int_C (x^2+y^2)\,dy$ C is the path from $(0,0)$ to $(1,0)$ and $(1,0)$ to $(1,3)$.



$C_1 \to (0,0)$ to $(1,0)$ $C_2 \to (1,0)$ to $(1,3)$

then: $$\int_C (x^2+y^2)\,dy = \int_{C_1} (x^2+y^2)\,dy + \int_{C_2} (x^2+y^2)\,dy$$

For $C_1 \to y=0$, $dy=0$

so, $$\int_{C_1} (x^2+y^2)\,dy = 0$$

For $C_2 \to x=1$, $0 \le y \le 3$

so, $$\int_{C_2} (x^2+y^2)\,dy = \int_{C_2} (1^2+y^2)\,dy$$

$$= \int_0^3 (1+y^2)\,dy$$

$$= \left[y+\frac{y^3}{3}\right]_0^3$$

$$= \left[3+\frac{3^3}{3}\right]_0^3$$

$$= 12$$

so, $$\int_C (x^2+y^2)\,dy = \int_{C_1} (x^2+y^2)\,dy + \int_{C_2} (x^2+y^2)\,dy$$

$$= 0 + 12$$

$$= 12$$

16.2.19 Find the work done by F over the curve in the direction of increasing t.

$F = 3xy\,i + 2y\,j - 3yz\,k \qquad r(t) = t\,i + t^2 j + t\,k \qquad 0 \le t \le 1$

$x = t \qquad y = t^2 \qquad z = t$

$F = 3 \cdot t \cdot t^2 i + 2t^2 j - 3 \cdot t^2 \cdot t\, k \qquad r'(t) = i + 2t\,j + k$

$= 3t^3 i + 2t^2 j - 3t^3 k$

$$\text{work} = \int_a^b F \cdot \frac{dr}{dt}\, dt = \int_a^b (3t^3 i + 2t^2 j - 3t^3 k) \cdot (i + 2t\,j + k)\, dt$$

$$= \int_0^1 (3t^3 + 4t^3 - 3t^3)\, dt$$

$$= \int_0^1 4t^3\, dt$$

$$= [t^4]_0^1$$

$$= 1$$

16.2.23 Evaluate $\int_C xy\, dx + (x+y)\, dy$ along the curve $y = 2x^2$ from $(-3, 18)$ to $(-2, 8)$.

$$\int_a^b (M\,dx + N\,dy) = \int_a^b \left(\overset{xy}{M} \frac{dx}{dt} + \overset{x+y}{N} \frac{dy}{dt} \right) dt$$

Parameterize

let $x = t \qquad y = 2t^2$

Thus,

$M = xy \qquad N = x + y$

$M = 2t^3 \qquad N = t + 2t^2$

$-3 \le t \le -2$ x limits

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 4t$

$$\int_C xy\, dx + (x+y)\, dy = \int_{-3}^{-2} \left(xy \frac{dx}{dt} + (x+y) \frac{dy}{dt} \right) dt$$

$$= \int_{-3}^{-2} \left(2t^3 \cdot 1 + (t + 2t^2)\, 4t \right) dt$$

$$= \int_{-3}^{-2} (2t^3 + 4t^2 + 8t^3)\, dt$$

$$= \int_{-3}^{-2} (10t^3 + 4t^2)\, dt$$

$$= \left[10\frac{t^4}{4} + 4\frac{t^3}{3} \right]_{-3}^{-2}$$

$$= \left[10\frac{(-2)^4}{4} + 4\frac{(-2)^3}{3} \right] - \left[10\frac{(-3)^4}{4} + 4\frac{(-3)^3}{3} \right] = \left(\frac{88}{3} - \frac{333}{2} \right) = -\frac{823}{6}$$

16.2.31 Find the circulation and flux of the field $F = -2xi - 2yj$ around and across the closed semicircular path that consists of the semicircular arch $r_1(t) = (a\cos t)i + (a\sin t)j$, $0 \le t \le \pi$ followed by the line segment $r_2(t) = ti$, $-a \le t \le a$.

$$Flow = \int_a^b F \cdot T\, ds = \int_{t=b}^{t=c} F \cdot T\, ds = \int_{t=b}^{t=c} f \cdot \frac{dr}{dt}\, dt = \int_0^{\pi} \left(F_1 \cdot \frac{dr_1}{dt}\right) dt + \int_{-a}^{a} \left(F_2 \cdot \frac{dr_2}{dt}\right) dt$$

$F_1 = -2xi - 2yj$

$= (-2a\cos t)i - (2a\sin t)j$ for $0 \le t \le \pi$

$F_2 = -2xi - 2yj$

$= -2ti$ for $-a \le t \le a$

$\frac{dr_1}{dt} = (-a\sin t)i + (a\cos t)j$ for $0 \le t \le \pi$

$\frac{dr_2}{dt} = i$ for $-a \le t \le a$

$F_1 \cdot \frac{dr_1}{dt} = (-2a\cos t\, i - 2a\sin t\, j) \cdot (-a\sin t\, i + a\cos t\, j)$

$= 2a^2 \sin t \cos t - 2a^2 \sin t \cos t$

$= 0$

$F_2 \cdot \frac{dr_2}{dt} = (-2ti) \cdot i$

$= -2t$

$\int_0^{\pi} 0\, dt = 0$

$\int_{-a}^{a} -2t\, dt = [-t^2]_{-a}^{a} = -a^2 - (-(-a)^2) = -a^2 + a^2 = 0$

$$\int_0^{\pi} \left(F_1 \cdot \frac{dr_1}{dt}\right) dt + \int_{-a}^{a} \left(F_2 \cdot \frac{dr_2}{dt}\right) dt = 0 \quad \text{circulation}$$

Flux of the field $F = Mi + Nj$ accross C

$Flux = \oint M\,dy - N\,dx$

$M_1 = -2x$

$= -2a\cos t$

$N_1 = -2y$

$= -2a\sin t$

$M_2 = -2x$

$= -2t$

$N_2 = -2y$

$= 0$

$dx_1 = (-a\sin t)dt$ $\quad dy_1 = (a\cos t)dt$ $\quad 0 \le t \le \pi$

$dx_2 = 1\,dt$ $\quad dy_2 = 0$ $\quad -a \le t \le a$

$\int_{C_1} M_1 dy_1 - N_1 dx_1 = \int_0^{\pi} (-2a\cos t(a\cos t) + 2a\cancel{\sin t}(-a\cancel{\sin t}))\, dt$

$= \int_0^{\pi} (-2a^2\cos^2 t - 2a^2\sin^2 t)\, dt$

$= \int_0^{\pi} (-2a^2)(\cos^2 t + \sin^2 t)\, dt$

$= \int_0^{\pi} -2a^2 \cdot 1\, dt$

$= [-2a^2 t]_0^{\pi}$

$= -2a^2\pi$

$= \int_{-a}^{a} (-2t \cdot 0 - 0 \cdot 1)\, dt = 0$

$-2a^2\pi + 0 = -2a^2\pi$

Find the circulation and flux of the field $\mathbf{F} = -7y\mathbf{i} + 7x\mathbf{j}$ around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (-p\cos t)\mathbf{i} + (-p\sin t)\mathbf{j}$, $0 \le t \le \pi$, followed by the line segment $\mathbf{r}_2(t) = -t\mathbf{i}$, $-p \le t \le p$.

The circulation is $7\pi p^2$. (Type an exact answer, using π as needed.)

The flux is 0. (Type an exact answer, using π as needed.)

The given vector field is $F = -7y\hat{i} + 7x\hat{j}$

Let C be the closed semicircular path that consists of the semicircular arc $r_1(t) = (-p\cos t)\hat{i} + (-p\sin t)\hat{j}$, $0 \le t \le \pi$ followed by the line segment $r_2(t) = -t\hat{i}$, $-p \le t \le p$.

Let S be the region bounded by the closed semicircular path C.

Now, the circulation is given by

$$\oint_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr \quad \text{---(1)}$$

where $C_1 : r_1(t) = (-p\cos t)\hat{i} + (-p\sin t)\hat{j}$, $0 \le t \le \pi$

and $C_2 : r_2(t) = -t\hat{i}$, $-p \le t \le p$.

Now, on the curve C_1, $x = -p\cos t$, $y = -p\sin t$, $0 \le t \le \pi$

$$\therefore \int_{C_1} F\cdot dr = \int_{C_1} (-7y\hat{i} + 7x\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_{C_1} -7y\,dx + 7x\,dy$$

$$= \int_0^{\pi} [(-7)(-p\sin t)(p\sin t) + 7(-p\cos t)(-p\cos t)]\,dt$$

$$= \int_0^{\pi} (7p^2\sin^2 t + 7p^2\cos^2 t)\,dt$$

$$= \int_0^{\pi} 7p^2(\sin^2 t + \cos^2 t)\,dt$$

$$= 7p^2 \int_0^{\pi} dt$$

$$= 7p^2(\pi - 0)$$

$$= 7\pi p^2$$

On the curve C_2, $x = -t$, $y = 0$, $-p \le t \le p$

Therefore $\int_{C_2} F\cdot dr = \int_{C_2} (-7y\,dx + 7x\,dy)$

$$= \int_{-p}^{p} -7t\cdot 0$$

$$= 0$$

From (1), we get

$$\oint_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr = 7\pi p^2 + 0 = 7\pi p^2$$

Therefore the required circulation is $7\pi p^2$.

The flux of F across S is given by

$$\iint_S F\cdot ds \quad \text{--- (2)}$$

On S, $n = \hat{k}$, $z = 0$ and $ds = dx\,dy$

$$\therefore \iint_S F\cdot ds = \iint_S F\cdot n\,ds$$

$$= \iint_S (-7y\hat{i} + 7x\hat{j})\cdot(\hat{k})\,dx\,dy$$

$$= \iint_S 0\,dx\,dy = 0$$

Therefore the required flux is 0.

16.2.45 Salt water with a density of $\delta = 0.25\ g/cm^2$ flows over the curve $r(t) = \sqrt{t}\,i + 3t\,j$, $0 \le t \le 4$, according to the vector field $F = \delta v$, where $v = xy\,i + (y - x)\,j$ is a velocity field measured in centimeters per second. Find the flow of F over the curve $r(t)$.

$$F = \delta v$$
$$= 0.25\,xy + 0.25(y - x)$$

$$F(r(t)) = 0.25\sqrt{t}\cdot 3t\,i + 0.25(3t - \sqrt{t})\,j$$
$$= 0.75t\sqrt{t}\,i + 0.75t - 0.25\sqrt{t}\,j$$

$$\frac{dr}{dt} = \left[\frac{d}{dt}(\sqrt{t})\right]i + \left[\frac{d}{dt}(3t)\right]j$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{t}}\,i + 3j \qquad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$F(r(t))\cdot\frac{dr}{dt} = (0.75t\sqrt{t}\,i + (0.75t - 0.25\sqrt{t})\,j)\cdot\left(\frac{1}{2\sqrt{t}}\,i + 3j\right)$$
$$= \frac{0.75t\cancel{\sqrt{t}}}{2\cancel{\sqrt{t}}} + 2.25t - 0.75\sqrt{t}$$
$$= 2.625 - 0.75\sqrt{t}$$

$$Flow = \int_a^b \left(F(r(t))\cdot\frac{dr}{dt}\right)dt = \int_0^4 (2.625t - 0.75\sqrt{t})\,dt = 17 \text{ grams per second.}$$

16.2.55 $F = 4xy\,i + 3y\,j + 2k$ is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve $r(t) = t\,i + t^2 j + k$, $0 \le t \le 3$ in the direction of increasing t.

$$F(r(t)) = 4 \cdot t \cdot t^2 i + 3t^2 j + 2k$$
$$= 4t^3 i + 3t^2 j + 2k$$

$$\frac{dr}{dt} = \frac{d}{dt}(r(t))$$
$$= \frac{d}{dt}(t\,i + t^2 j + k)$$
$$= i + 2t\,j$$

$$F(r(t)) \cdot \frac{dr}{dt} = (4t^3 i + 3t^2 j + 2k) \cdot (i + 2t\,j)$$
$$= 4t^3 + 6t^3$$
$$= 10t^3$$

$$\int_{t=a}^{t=b} F \cdot T\, ds = \int_a^b F \cdot \frac{dr}{dt}\, dt = \int_0^3 10t^3\, dt$$
$$= \left[\frac{5}{2}t^4\right]_0^3$$
$$= \frac{405}{2}$$

Path Independence, Conservative Fields, and Potential Functions.

16.3.1 Determine if the field $F = 16yz\,i + 16xz\,j + 16xy\,k$ is conservative or not conservative.

Let $F = M(x,y,z)\,i + N(x,y,z)\,j + P(x,y,z)\,k$ be the field.

F is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\frac{\partial P}{\partial y} = 16x$ ✓ $\frac{\partial N}{\partial z} = 16x$

$\frac{\partial M}{\partial z} = 16y$ ✓ $\frac{\partial P}{\partial x} = 16y$

$\frac{\partial N}{\partial x} = 16z$ ✓ $\frac{\partial M}{\partial y} = 16z$

$F = 16yz\,i + 16xz\,j + 16xy\,k$ is conservative

16.3.3 Determine if the field $F = 8y\,i + 8(x+z)\,j - 8y\,k$ is conservative or not conservative.

$M = 8y\,i$ $N = 8(x+z)\,j$ $P = -8y\,k$

$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\frac{\partial P}{\partial y} = -8$

$\ne$ thus, F is not conservative

$\frac{\partial N}{\partial z} = 8$

16.3.7 Find the potential function f for the field $\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 9z\mathbf{k}$

$$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = 4y \qquad \frac{\partial f}{\partial z} = 9z$$

Integrate $\frac{\partial f}{\partial x}$

$\int 4x \, dx = 2x^2 + g(y,z)$ → arbitrary constant

$$f = 2x^2 + g(y,z)$$

Partially differentiate $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} \rightarrow \frac{\partial g}{\partial y} = 4y$$

Integrate $\frac{\partial g}{\partial y}$

$\int 4y \, dy = 2y^2 + h(z)$ → arbitrary constant

$$g = 2y^2 + h(z)$$

Join f and g

$$f = 2x^2 + 2y^2 + h(z)$$

Partially differentiate $\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial z} \rightarrow \frac{\partial h}{\partial z} = 9z$$

Integrate $\frac{\partial g}{\partial y}$

$$\int 9z \, dz = \frac{9}{2}z^2 + C$$

$$h = \frac{9}{2}z^2 + C$$

Join f, g and h

$$f = 2x^2 + 2y^2 + \frac{9}{2}z^2 + C$$

16.3.9 Find the potential function F for the field $F = e^{3y+4y}(2i + 6xj + 8xk)$

$F = e^{3y+4z}(2i + 6xj + 8xk)$.

$F = 2e^{3y+4z}i + 6xe^{3y+4z}j + 8xe^{3y+4z}k$.

$F_x = 2e^{3y+4z}$, $F_y = 6xe^{3y+4z}$, $F_z = 8xe^{3y+4z}$

$F_x = 2e^{3y+4z} \Rightarrow F = 2xe^{3y+4z} + g(y,z)$

$F_y = (2xe^{3y+4z} \times 3) + g'(y,z) = 6xe^{3y+4z}$

$\Rightarrow 6xe^{3y+4z} + g'(y,z) = 6xe^{3y+4z}$

$\Rightarrow g'(y,z) = 0$.

$\Rightarrow g(y,z) = g(z)$

$F_y = 2xe^{3y+4z} + g(z)$

$F_z = (2xe^{3y+4z} \times 4) + g'(z) = 8xe^{3y+4z}$

$\Rightarrow 8xe^{3y+4z} + g'(z) = 8xe^{3y+4z}$

$\Rightarrow g'(z) = 0 \Rightarrow g(z) = C$

$\therefore F = 2xe^{3y+4z} + C$ is the potential function

16.3.13 Show that the differential form in the integral below is exact.
Then evaluate the integral.

$$\int_{(0,0,0)}^{(4,-2,-5)} 12x\,dx + 16y\,dy + 10z\,dz$$

the differential form $M\,dx + N\,dy + P\,dz$ is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z};\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x};\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

use : $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z};\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x};\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$M = 12x \quad N = 16y \quad P = 10z$

$\frac{\partial P}{\partial y} = 0 \quad \frac{\partial M}{\partial z} = 0 \quad \frac{\partial N}{\partial x} = 0$

✓ ✓ ✓

$\frac{\partial N}{\partial z} = 0 \quad \frac{\partial P}{\partial x} = 0 \quad \frac{\partial M}{\partial y} = 0$

the differential form $12x\,dx + 16y\,dy + 10z\,dz$ is exact.

$f(x,y,z) = \int 12x\,dx = 6x^2 + g(y,z)$

$g(y,z) = \int 16y\,dy = 8y^2 + h(z)$

$h(z) = \int 10z\,dz = 5z^2 + C$

$$f(x,y,z) = \int_{(0,0,0)}^{(4,-2,-5)} 12x\,dx + 16y\,dy + 10z\,dz = \Big[6x^2 + 8y^2 + 5z^2 + C\Big]_{(0,0,0)}^{(4,-2,-5)}$$

$$= 253$$

16.3.23 Evaluate the integral $\int_{(4,2,4)}^{(8,4,-1)} y\,dx + x\,dy + 4\,dz$ by finding parametric equations for the line segment from $(4,2,4)$ to $(8,4,-1)$ and evaluating the line integral of $F = yi + xj + 4k$ along the segment. Since F is conservative, the integral is independent of the path.

line integral

$$\int_{t=a}^{t=b} F \cdot T\,ds = \int_a^b F \cdot \frac{dr}{dt}\,dt$$

$F = yi + xj + 4k$

$(x_1, y_1, z_1) = (4, 2, 4)$

$(x_2, y_2, z_2) = (8, 4, -1)$

parametric equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

$$\frac{x-4}{8-4} = \frac{y-2}{4-2} = \frac{z-4}{-1-4} = t$$

thus, $x = 4t + 4$ $\quad y = 2t + 2$ $\quad z = -5t + 4$

$dx = 4\,dt$ $\quad dy = 2\,dt$ $\quad dz = -5\,dt$

at $(4, 2, 4)$, $t = 0$

at $(8, 4, -1)$, $t = 1$

$$\therefore \quad F(r(t)) = yi + xj + 4k$$
$$= (2t+2)i + (4t+4)j + 4k$$

$$F(r(t)) \cdot \frac{dr}{dt} = (2t+2)4\,dt + (4t+4)2\,dt + 4(-5\,dt)$$
$$= (8t + 8 + 8t + 8 - 20)\,dt$$
$$= (16t - 4)\,dt$$

$$\int_{t=a}^{t=b} F \cdot T\,ds = \int_a^b F \cdot \frac{dr}{dt}\,dt = \int_0^1 (16t - 4)\,dt$$
$$= \left[8t^2 - 4t\right]_0^1$$
$$= 4$$

16.3.25 Show that the value of the integral below does not depend on the path taken from A to B.

$$\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$$

Compare the expression $M\,dx + N\,dy + P\,dz$

$M = z^2 \quad N = 2y \quad P = 2xz$

$\frac{\partial P}{\partial y} = 0 \quad \frac{\partial M}{\partial z} = 2z \quad \frac{\partial N}{\partial x} = 0$

$\frac{\partial N}{\partial z} = 0 \quad \frac{\partial P}{\partial x} = 2z \quad \frac{\partial M}{\partial y} = 0$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z};\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x};\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$ ✓

therefore, the function $F = z^2dx + 2y\,dy + 2xz\,dz$ is conservative.

so, its integration is path independent.

$$\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz = \left[0 + y^2 + 2x \cdot \frac{z^2}{2}\right]_A^B$$
$$= \left[xz^2 + y^2\right]_A^B$$

16.3.27 Find a potential function for **F**.

$$\mathbf{F} = \frac{2x}{y}i + \frac{10-x^2}{y^2}j \quad \{(x,y): y>0\}$$

$$M = \frac{2x}{y}i \qquad N = \frac{10-x^2}{y^2}j$$

$$\frac{\partial f}{\partial x} = \frac{2x}{y} \qquad \frac{\partial f}{\partial y} = \frac{10-x^2}{y^2}$$

$$f(x,y) = \int \frac{2x}{y}\,dx = \frac{x^2}{y} + g(y)$$

$$x^2 \cdot y^{-1} = -x^2y^{-2} = -\frac{x^2}{y^2}$$

$$\frac{\partial f}{\partial y} = \cancel{\frac{-x^2}{y^2}} + g'(y) = \frac{10-\cancel{x^2}}{\cancel{y^2}}$$

$$g'(y) = \frac{10}{y^2}$$

$$\frac{10}{y^2} = 10 \cdot \frac{y^{-2+1}}{-2+1} = -10y^{-1} = \frac{-10}{y}$$

$$g(y) = \int \frac{10}{y^2}\,dy = \frac{-10}{y}$$

$$f(x,y) = \frac{x^2}{y} - \frac{10}{y} + c$$

$$f(x,y,z) = \int_{(0,0,0)}^{(4,-2,-5)} 12x\,dx + 16y\,dy + 10z\,dz = \Big[6x^2 + 8y^2 + 5z^2 + C\Big]_{(0,0,0)}^{(4,-2,-5)}$$

$$= 253$$

Green's Theorem in the Plane

16.4.3 Find the k-component of (curl F) for the following vector field on the plane.

$$\mathbf{F} = (xe^y)i + (9ye^x)j$$

$$(\text{curl } F)\cdot k = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= \frac{\partial N}{\partial x} 9ye^x - \frac{\partial M}{\partial y} xe^y$$

$$= 9ye^x - xe^y$$

16.4.1 Find the k-component of (curl F) for the following vector field on the plane.

$$\mathbf{F} = (x+4y)i + (9xy)j$$

$$(\text{curl } F)\cdot k = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= \frac{\partial N}{\partial x} 9xy - \frac{\partial M}{\partial y} x+4y$$

$$= 9y - 4$$

16.4.11 Use Green's Theorem to find the counterclockwise circulation and outward flux for the field

$\mathbf{F} = (8x - y)\mathbf{i} + (2y - x)\mathbf{j}$ and curve C: the square bounded by $x = 0$, $x = 9$, $y = 0$, $y = 9$.

Flux divergence

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

Find $\frac{\partial M}{\partial x}$ and $\frac{\partial N}{\partial y}$

$\frac{\partial M}{\partial x} = (8x - y)$
$= 8$

$\frac{\partial N}{\partial y} = (2y - x)$
$= 2$

thus, $\int_0^9 \int_0^9 10\,dx\,dy = \int_0^9 \left[10x\right]_0^9 dy$

$= \int_0^9 90\,dy$

$= \left[90y\right]_0^9 = 810$ The flux is 810

Circulation Curl

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Find $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$

$\frac{\partial N}{\partial x} = (2y - x)$
$= -1$

$\frac{\partial M}{\partial y} = (8x - y)$
$= -1$

thus, $\iint_R 0\,dx\,dy$ The circulation is 0.

16.4.13 Use Green's Theorem to find the counterclockwise circulation and outward flux for the field

$\mathbf{F} = (y^2 - 5x^2)\mathbf{i} + (5x^2 + y^2)\mathbf{j}$ and curve C: the square bounded by $y = 0$, $x = 3$, and $y = x$.

$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = (y^2 - 5x^2) + (5x^2 + y^2)$
$= -10x + 2y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (5x^2 + y^2) - (y^2 - 5x^2)$
$= 10x - 2y$

Flux divergence

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{dy}\right) dx\,dy$$

$= \int_0^x \int_0^3 (-10x + 2y)\,dx\,dy$

Integrate with respect to y first.

$= \int_0^3 \int_0^x (-10x + 2y)\,dy\,dx$

$= \int_0^3 \left[-10xy + y^2\right]_0^x dx$

$= \int_0^3 (-10x^2 + x^2)\,dx$

$= \int_0^3 -9x^2\,dx$

$= \left[-3x^3\right]_0^3$

$= -81$

Circulation curl

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{dy}\right) dx\,dy$$

$= \int_0^x \int_0^3 (10x - 2y)\,dx\,dy$

Integrate with respect to y first.

$= \int_0^3 \int_0^x (10x - 2y)\,dy\,dx$

$= \int_0^3 \left[10xy - y^2\right]_0^x dx$

$= \int_0^3 (10x^2 - x^2)\,dx$

$= \int_0^3 9x^2\,dx$

$= \left[3x^3\right]_0^3$

$= 81$

16.4.25 Find the work done by $F = 4xy^3 i + 7x^2y^2 j$ in moving a particle once counterclockwise around the curve C: the boundary of the "triangular" region in the first quadrant enclosed by the x-axis, the line $x = 1$, and the curve $y = x^3$.

Circulation curl

$$\oint_C F \cdot T\, ds = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (7x^2y^2) - (4xy^3)$$
$$= 14xy^2 - 12xy^2$$
$$= 2xy^2$$

$$\oint_C F \cdot T\, ds = \int_0^{x^3}\int_0^1 2xy^2\, dx\, dy$$

Integrate with respect to y first.

$$= \int_0^1\int_0^{x^3} 2xy^2\, dy\, dx$$

$$= \int_0^1 \left[\frac{2}{3}xy^3\right]_0^{x^3} dx$$

$$= \int_0^1 \left(\frac{2}{3}x^{10}\right) dx$$

$$= \left[\frac{2}{33}x^{11}\right]_0^1$$

$$= \frac{2}{33}$$

16.4.29 Use Green's Theorem to evaluate the integral.

$$\oint (2y + x)\, dx\, dy + (y + 7x)\, dy \qquad C: \text{The circle } (x-7)^2 + (y-2)^2 = 2$$

$$\oint_C F \cdot T\, ds = \oint_C M dx + N dy \overset{2}{=} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (y + 7x) - (2y + x)$$
$$= 7 - 2$$
$$= 5$$

$$\oint_C F \cdot T\, ds = \iint_R 5\, dx\, dy$$

$$= 5 \cdot 2\pi$$ since area of a circle is πr^2 and r^2 is 2. $\therefore$ $A = 2\pi$

$$= 10\pi$$

16.4.34

Use the Green's Theorem area formula shown on the right to find the area of the region enclosed by the given curves.

One arch of the cycloid $x = 4t - 4\sin t$, $y = 4 - 4\cos t$ and the x-axis.

Green's Theorem Area Formula

Area of R $= \frac{1}{2}\oint_C x\,dy - y\,dx$

Consider one arch of the cycloid $x = 4t - 4\sin t$, $y = 4 - 4\cos t$ and the x-axis.
Then,

$$dx = (4 - 4\cos t)\,dt, \quad dy = 4\sin t\,dt.$$

The area is bounded by the x-axis on the bottom from $x = 0$ to $x = 2\pi$ and by the cycloid on the top.

Use Green's Theorem, the area of the region R enclosed by the above curves is,

$$\begin{aligned}
A &= \frac{1}{2}\oint_C x\,dy - y\,dx \\
&= \frac{1}{2}\int_0^{2\pi}\left[(4t - 4\sin t)(4\sin t\,dt) - (4 - 4\cos t)(4 - 4\cos t)\,dt\right] \\
&= \frac{1}{2}\int_0^{2\pi}\left[(16t\sin t - 16\sin^2 t) - (16 - 32\cos t + 16\cos^2 t)\right]dt \\
&= \frac{1}{2}\int_0^{2\pi}(16t\sin t - 16\sin^2 t - 16 + 32\cos t - 16\cos^2 t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi}(16t\sin t - 16 - 16 + 32\cos t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi}(16t\sin t - 32 + 32\cos t)\,dt \\
&= \frac{16}{2}\int_0^{2\pi}(t\sin t - 2 + 2\cos t)\,dt \\
&= 8(-t\cos t + \sin t - 2t + 2\sin t)_0^{2\pi} \\
&= 8\left[(-2\pi + 0 - 4\pi + 0) - (0 + 0 - 0 + 0)\right] \\
&= 8(-6\pi) \\
&= -48\pi
\end{aligned}$$

Since the area is a positive quantity.
Thus, the area of the region is $\boxed{48\pi}$.